

# Models of Quantum Environment

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## Introduction

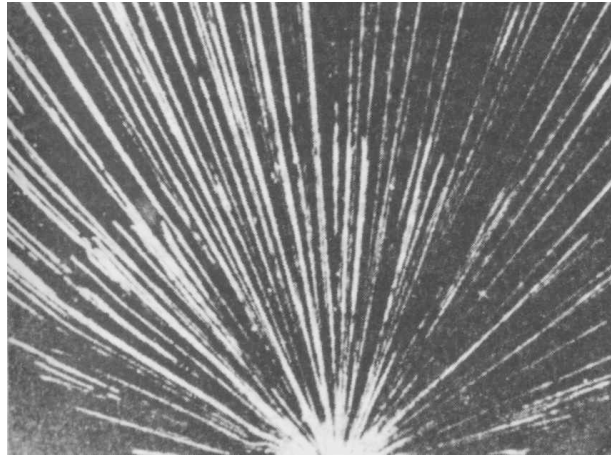
Aim: to investigate the role of interaction and entanglement in the appearance of classical behaviour in quantum systems

Motivations:

- relevance in the context of foundational problems of QM ("Quantum Mechanics without collapse")
- analysis of timescales of quantum decoherence phenomena
- modelling mesoscopic systems
- exploration of the border quantum-classical

Methodology: simplified models of many-components quantum system

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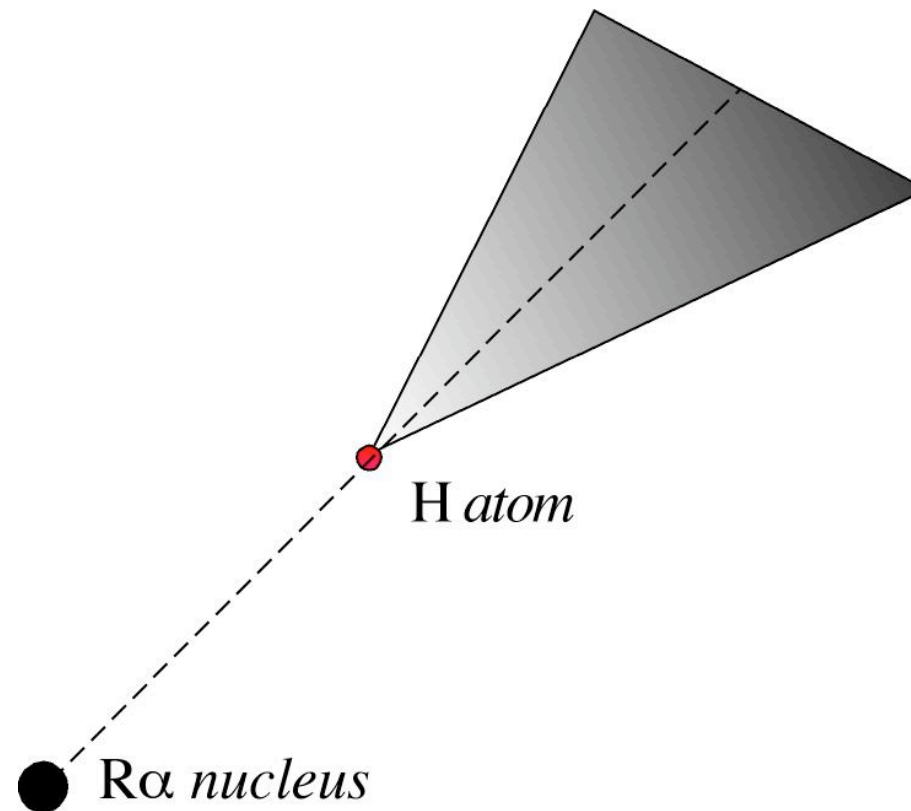


- Emission of alpha particles by a radiating atom inside a Wilson chamber<sup>†</sup>
  - According to the Gamow's theory of radioactive decay an alpha particle is well described by a spherical wave function
  - *"It is a little difficult to picture how it is that an outgoing spherical wave can produce a straight track; we think intuitively that it should ionise atoms at random throughout space"*<sup>†</sup>
- Qualitative description of the phenomenon
  - The ionization of atoms along the "trajectory" of the particle produces a local phase transition
  - Long range correlations guarantee the amplification of the effect toward a macroscopic scale

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<sup>†</sup> N. F. Mott, *Proc. Roy. Soc. A*, **126**, 79-84, 1929.

- Mott's results
    - Mott considers a system made up of one alpha particle and two atoms
    - Initially the atoms are in the ground state and the state of the particle is an outgoing spherical wave packet
    - Using time independent perturbation theory he reaches the conclusion that the probability of a double ionization is maximal if the two atoms and the origin are on a straight line
    - Intuitive conclusion: when in presence of a spherically symmetric array of atoms, the wave function very quickly becomes the sum of terms supported on disjoint regions of the global configuration space
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- The work of Mott can be viewed as the first attempt to give an explanation (thoroughly inside the framework of Quantum Mechanics) of the transition "quantum to classical" as a consequence of the diffusion of entanglement to larger and larger portions of the environment

## **Are there "solvable" interesting macroscopic Quantum Systems?**

- To build up models of Quantum Environments or Measurement Apparatuses one needs specific results on the dynamics of multicomponent quantum systems
- Only trivial  $N$  particle quantum systems are explicitly solvable
- Even systems made up of one test particle in interaction with a large number of non-interacting quantum particles are not solvable
- There is no conceivable "classical behaviour" of a discrete quantum system (e.g. a spin)

Particles interacting with arrays of discrete quantum system seems to be the only tenable choice

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## An already difficult simple model

- A three particle one-dimensional model

$$\psi = \psi(R, r_1, r_2; t) \quad R, r_i \in \mathbb{R}; \quad t \in \mathbb{R}^+$$

$$i\hbar \frac{d}{dt} \psi = H_0 \psi + H_1 \psi$$

$$H_0 = -\frac{\hbar^2}{2M} \Delta_R + \sum_{i=1,2} \left( -\frac{\hbar^2}{2m} \Delta_{r_i} + \frac{1}{2} \omega^2 (r_i - a_i)^2 \right); \quad H_1 = \lambda \sum_{i=1,2} \delta(R - r_i)$$

$$\psi(R, r_1, r_2; 0) = \phi(R) \varphi_0(r_1 - a_1) \varphi_0(r_2 - a_2)$$

$$\phi(R) = \frac{\mathcal{N}}{\sqrt{\sigma}} e^{-\frac{R^2}{2\sigma^2}} e^{+i \frac{P_0}{\hbar} R} + \frac{\mathcal{N}}{\sqrt{\sigma}} e^{-\frac{R^2}{2\sigma^2}} e^{-i \frac{P_0}{\hbar} R}$$

$$\varphi_0(x) = \frac{e^{-\frac{x^2}{2\gamma^2}}}{\pi^{1/4} \sqrt{\gamma}} \quad \gamma = \sqrt{\frac{\hbar}{m\omega}}$$

- Time-dependent perturbation theory

$$\psi(t) = e^{-i\frac{t}{\hbar}H_0}\psi_0 - \frac{i}{\hbar} \int_0^t ds e^{-i\frac{(t-s)}{\hbar}H_0} H_1 \psi(s)$$

- Zero-th order

$$\psi^{(0)}(t) = e^{-i\frac{t}{\hbar}H_0}\psi_0$$

- First order

$$\psi^{(1)}(t) = e^{-i\frac{t}{\hbar}H_0}\psi_0 - \frac{i}{\hbar} \int_0^t ds e^{-i\frac{(t-s)}{\hbar}H_0} H_1 e^{-i\frac{s}{\hbar}H_0}\psi_0$$

- Second order

$$\psi^{(2)}(t) = e^{-i\frac{t}{\hbar}H_0}\psi_0 - \frac{i}{\hbar} \int_0^t ds e^{-i\frac{(t-s)}{\hbar}H_0} H_1 \psi^{(1)}(s)$$

Expanding in Hermite polynomials

$$\psi^{(2)}(R, r_1, r_2; t) \equiv \sum_{j_1, j_2} f_{j_1, j_2}^{(2)}(R; t) \varphi_{j_1}(r_1 - a_1) \varphi_{j_2}(r_2 - a_2)$$


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- Under the assumptions

$$\delta m \equiv \frac{m}{M} \ll 1, \quad \delta E \equiv \frac{\hbar\omega}{Mv_0^2} \ll 1,$$

$$\delta\tau \equiv \frac{v_0}{\omega|a_j|} \ll 1$$

$$\delta R \equiv \frac{\sigma}{|a_j|} \ll 1,$$

$$\delta V \equiv \frac{\lambda}{Mv_0^2} \ll 1, \quad \delta L \equiv \frac{\delta}{|a_j|} \ll 1$$

the probability that a double ionization occurs is much higher if the initial momentum is heading toward the atoms (!!!). The interaction with the first oscillator in  $a_1 > 0$  produces a wave packet with  $p > 0$  and this enhances the excitation probability of the second oscillator only if  $a_2 > a_1$ .

The proof is based on a stationary (and non-stationary) phase analysis for  $f_{j_1, j_2}^{(2)}(R; t)$

d=1, G.F. Dell'Antonio, R.F., A.Teta - to appear in JMP **49**; d=3 in preparation

## A particle in an array of spins

- The Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes (\mathbb{C}^2)^{\otimes N}$$

$$\Psi \in \mathcal{H} \quad \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}}(x) \otimes \chi_{\underline{\sigma}} \quad \underline{\sigma} = \{\sigma_1, \dots, \sigma_N\} \quad \sigma_j = \pm$$

$$\chi_{\underline{\sigma}} = \chi_{\sigma_1} \otimes \cdots \otimes \chi_{\sigma_N} \in (\mathbb{C}^2)^{\otimes N} \quad \hat{\sigma}_{jx} \chi_{\sigma_j} = \sigma_j \chi_{\sigma_j} \quad \hat{\sigma}_{jx} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\|\chi_{\sigma_j}\|_{\mathbb{C}^2} = 1$$

The scalar product in  $\mathcal{H}$  is

$$\langle \Psi_1, \Psi_2 \rangle = \sum_{\underline{\sigma}} (\psi_{1\underline{\sigma}}, \psi_{2\underline{\sigma}})_{L^2}$$

- Define the operator on  $(\mathbb{C}^2)^{\otimes N}$

$$\mathbf{s}_j = \overbrace{\mathbb{I}_{\mathbb{C}^2} \otimes \cdots \otimes \hat{\sigma}_{jx} \otimes \cdots \otimes \mathbb{I}_{\mathbb{C}^2}}^N \quad \mathbf{s}_j \chi_{\underline{\sigma}} = \sigma_j \chi_{\underline{\sigma}}$$

*j-th*

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- The “free” Hamiltonian

$$D(H) = H^2(\mathbb{R}^3) \otimes (\mathbb{C}^2)^{\otimes N}$$

$$H = -\Delta \otimes \mathbb{I}_{(\mathbb{C}^2)^{\otimes N}} + \sum_{j=1}^N \mathbb{I}_{L^2} \otimes \alpha_j \mathbf{s}_j \quad \alpha_j \in \mathbb{R}$$

$$H\Psi = \sum_{\underline{\sigma}} \left( -\Delta + \underline{\alpha} \underline{\sigma} \right) \psi_{\underline{\sigma}}(x) \otimes \chi_{\underline{\sigma}} ; \quad \underline{\alpha} = \{\alpha_1, \dots, \alpha_N\} ; \quad \underline{\alpha} \underline{\sigma} = \sum_{j=1}^N \alpha_j \sigma_j$$

- Spectrum of  $H$

$$\sigma_{pp} = \emptyset \quad \sigma_{ess} = \sigma_{ac} = [\mu, \infty) \quad \mu = \min_{\underline{\sigma}} (\underline{\alpha} \underline{\sigma})$$

- Resolvent of  $H$ ,  $R(z) = (H - z)^{-1}$

$$R(z)\Psi = \sum_{\underline{\sigma}} \left( -\Delta - (z - \underline{\alpha} \underline{\sigma}) \right)^{-1} \psi_{\underline{\sigma}}(x) \otimes \chi_{\underline{\sigma}} \quad z \in \rho(H)$$

where  $\rho(H)$  indicates the resolvent set of  $H$ .

- Point perturbations of  $H$

- Symmetric operator

$$D(H_0) = C_0^\infty(\mathbb{R}^3 \setminus \underline{y}) \otimes (\mathbb{C}^2)^{\otimes N} \quad \underline{y} = \{y_1, \dots, y_N\}$$

$$H_0 = -\Delta \otimes \mathbb{I}_{(\mathbb{C}^2)^{\otimes N}} + \sum_{j=1}^N \mathbb{I}_{L^2} \otimes \alpha_j \mathbf{s}_j \quad \alpha_j \in \mathbb{R}$$

- Deficiency indices

$$(H_0^* - z)\Phi^z = 0 \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad \Phi^z \in \mathcal{H} \quad (1)$$

$$\Phi_{j\sigma}^z = G^{z-\alpha\sigma}(x - y_j) \otimes \mathcal{X}_\sigma$$

$$G^z(x) = \frac{e^{i\sqrt{z}|x|}}{4\pi|x|} \quad z \in \mathbb{C} \setminus \mathbb{R}^+ \quad \text{Im}(\sqrt{z}) > 0$$

- There are  $N2^N$  independent solutions of equation (1) both for  $\text{Im}z > 0$  that for  $\text{Im}z < 0$ , then the deficiency indices are  $(n_+, n_-) = (N2^N, N2^N)$

## A pedagogical example: $N=1$ , $d=3$

The “free” Hamiltonian

$$D(\hat{H}_0) := \left\{ \Psi \in \mathcal{H} \left| \Psi = \Psi^z + \sum_{\sigma} q_{\sigma} \Phi_{\sigma}^z; \Psi^z = \sum_{\sigma} \psi_{\sigma}^z \otimes \chi_{\sigma} \in D(H); z \in \rho(\hat{H}_0); \right. \right. \\ \left. \left. \alpha q_{\sigma} = f_{\sigma}, d = 3 \right\} \quad f_{\sigma} = \lim_{|x| \rightarrow 0} \left[ \psi_{\sigma}(x) - \frac{q_{\sigma}}{4\pi |x|} \right] ; \alpha < 0$$

$$\hat{H}_0 \Psi := H \Psi^z + z \sum_{\sigma} q_{\sigma} \Phi_{\sigma}^z \quad \Psi \in D(\hat{H}_0).$$

With resolvent

$$\hat{R}_0(z) = R(z) + \sum_{\sigma, \sigma'} ((\Gamma_0(z))^{-1})_{\sigma, \sigma'} \langle \Phi_{\sigma'}^{\bar{z}}, \cdot \rangle \Phi_{\sigma}^z \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where

$$\Gamma_0(z) = \begin{pmatrix} \frac{\sqrt{z - \beta}}{4\pi i} + \alpha & 0 \\ 0 & \frac{\sqrt{z + \beta}}{4\pi i} + \alpha \end{pmatrix} \quad d = 3.$$


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- $\hat{H}_0$  is a two independent channel (attractive) point interaction Hamiltonian with thresholds respectively in  $+$  and  $-\beta$
  - with a suitable choice of  $\beta$  and  $\alpha$  the eigenvalue  $\beta - (4\pi\alpha)^2$  is embedded in the continuous part of the spectrum while the ground state is isolated and has an energy  $-\beta - (4\pi\alpha)^2$
  - the particle does not affect the dynamics of the spin
  - if the particles are many (non interacting among them) and the localized scatterer has many discrete energy levels the Hamiltonian can exhibit any number of isolated bound states, embedded eigenvalues and thresholds
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### The perturbed Hamiltonian

$$D(\hat{H}_\varepsilon) := \left\{ \Psi \in \mathcal{H} \left| \Psi = \Psi^z + \sum_{\sigma} q_{\sigma} \Phi_{\sigma}^z; \Psi^z = \sum_{\sigma} \psi_{\sigma}^z \otimes \chi_{\sigma} \in D(H); z \in \rho(\hat{H}_\varepsilon); \right. \right. \\ \left. \left. \alpha q_{\pm} + \varepsilon q_{\mp} = f_{\pm} \quad d = 3 \right\}$$

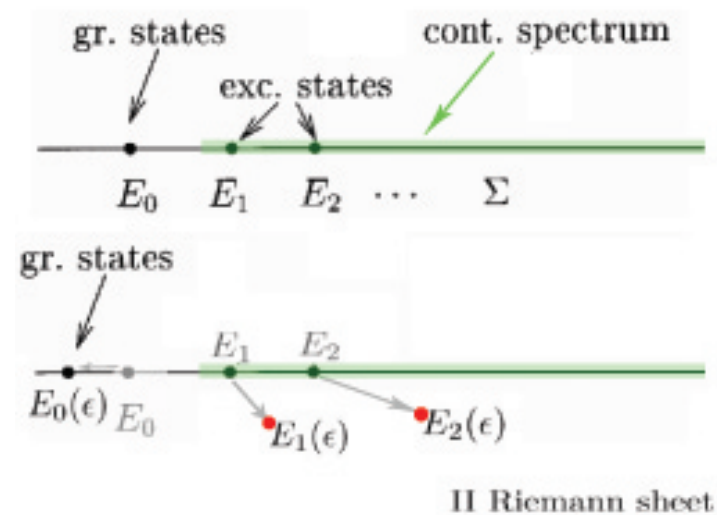
$$\hat{H}_\varepsilon \Psi := H \Psi^z + z \sum_{\sigma} q_{\sigma} \Phi_{\sigma}^z \quad \Psi \in D(\hat{H}_\varepsilon).$$

$$\hat{R}_\varepsilon(z) = R(z) + \sum_{\sigma, \sigma'} ((\Gamma_\varepsilon(z))^{-1})_{\sigma, \sigma'} \langle \Phi_{\sigma'}^{\bar{z}}, \cdot \rangle \Phi_{\sigma}^z \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where

$$\Gamma_\varepsilon(z) = \begin{pmatrix} -\frac{i\sqrt{z-\beta}}{4\pi} + \alpha & \varepsilon \\ \varepsilon & -\frac{i\sqrt{z+\beta}}{4\pi} + \alpha \end{pmatrix} \quad d = 3$$

- The spectral structure of the perturbed Hamiltonian
  - the ground state has moved slightly toward lower energies
  - the embedded eigenvalue moves into the second Riemann sheet giving rise to a resonance with position and width explicitly computable at any order in  $\epsilon$



- a very effective toy model for an atom coupled to the radiation field



## Toward a tracking chamber

- The “free” Hamiltonian for  $N=2$  in  $d=3$

$$D(H) = H^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$H = -\Delta + \alpha(\hat{\sigma}_{1x} + \hat{\sigma}_{2x}) \quad \alpha > 0$$

$$H\Psi = \sum_{\underline{\sigma}} (-\Delta + \underline{\alpha} \underline{\sigma}) \psi_{\underline{\sigma}}(x) \otimes \chi_{\underline{\sigma}} \quad \underline{\alpha} \underline{\sigma} = \alpha(\sigma_1 + \sigma_2)$$

- The spectrum of  $H$

$$\sigma_{pp}(H) = 0 \quad \sigma_{ess}(H) = \sigma_{ac}(H) = [-2\alpha, \infty)$$

- The unitary group  $e^{-iHt}$

$$\left\{ \begin{array}{l} i \frac{d\Psi^t}{dt} = H\Psi^t \\ \Psi^0 = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}}^0(x) \otimes \chi_{\underline{\sigma}} \end{array} \right. \Rightarrow \Psi^t = e^{-iHt} \Psi^0 = \sum_{\underline{\sigma}} (U^t \psi_{\underline{\sigma}}^0)(x) \otimes e^{-i\underline{\alpha} \underline{\sigma} t} \chi_{\underline{\sigma}}$$

where  $U^t = e^{i\Delta t}$

- The perturbed Hamiltonian can be defined through its domain and action

$$D(\hat{H}) = \left\{ \Psi \in \mathcal{H} : \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}}(x) \otimes \mathcal{X}_{\underline{\sigma}} = \Psi^z + \sum_{j, \underline{\sigma}} q_{j\underline{\sigma}} \frac{e^{i\sqrt{z-\underline{\alpha}\underline{\sigma}}|x-y_j|}}{4\pi|x-y_j|} \otimes \mathcal{X}_{\underline{\sigma}}; \right.$$

$$\Psi^z \in D(H), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad \text{Im}\sqrt{z-\underline{\alpha}\underline{\sigma}} > 0, \quad q_{j\underline{\sigma}} \in \mathbb{C},$$

$$\lim_{|x-y_1| \rightarrow 0} \left( \psi_{(\pm, \sigma_2)}(x) - \frac{q_{1(\pm, \sigma_2)}}{4\pi|x-y_1|} \right) = \pm i\gamma q_{1(\mp, \sigma_2)}$$

$$\lim_{|x-y_2| \rightarrow 0} \left( \psi_{(\sigma_1, \pm)}(x) - \frac{q_{2(\sigma_1, \pm)}}{4\pi|x-y_2|} \right) = \pm i\gamma q_{2(\sigma_1, \mp)}, \quad \gamma > 0 \left\}$$

$$\hat{H}\Psi = H\Psi^z + z \sum_{j, \underline{\sigma}} q_{j\underline{\sigma}} \frac{e^{i\sqrt{z-\underline{\alpha}\underline{\sigma}}|x-y_j|}}{4\pi|x-y_j|} \otimes \mathcal{X}_{\underline{\sigma}} \quad \Psi \in D(\hat{H})$$

- Or alternative via its resolvent

$$R_\gamma(z) := R(z) + \sum_{j, \underline{\sigma}, j', \underline{\sigma}'} (\Gamma_\gamma(z))_{j\underline{\sigma}, j'\underline{\sigma}'}^{-1} \langle \Phi_{j'\underline{\sigma}'}^{\bar{z}}, \cdot \rangle \Phi_{j\underline{\sigma}}^z; \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

where

$$\Phi_{j\underline{\sigma}}^z = G^{z-\underline{\sigma}\alpha}(\cdot - y_j) \otimes \mathcal{X}_{\underline{\sigma}}; \quad j = 1, 2 \quad \underline{\sigma} = (\sigma_1, \sigma_2), \quad \sigma_1, \sigma_2 = \pm,$$

$$(\Gamma_\gamma(z))_{j\underline{\sigma}, j'\underline{\sigma}'} = 0 \quad j \neq j' ; \underline{\sigma} \neq \underline{\sigma}'$$

$$(\Gamma_\gamma(z))_{j\underline{\sigma}, j'\underline{\sigma}} = -G^{z-\underline{\sigma}\alpha}(y_j - y_{j'}) \quad j \neq j'$$

$$(\Gamma_\gamma(z))_{j\underline{\sigma}, j\underline{\sigma}'} = 0 \quad \sigma_k \neq \sigma'_k \text{ for } k \neq j$$

$$(\Gamma_\gamma(z))_{j\underline{\sigma}, j\underline{\sigma}'} = \sigma'_j i\gamma \quad \sigma'_j \neq \sigma_j \text{ and } \sigma_k = \sigma'_k \text{ for } k \neq j$$

$$(\Gamma_\gamma(z))_{j\underline{\sigma}, j\underline{\sigma}} = \frac{\sqrt{z - \underline{\sigma}\alpha}}{4\pi i}$$

and the link with the previous definition is given by

$$q_{j\underline{\sigma}} = \sum_{j', \underline{\sigma}'} (\Gamma_\gamma(z))_{j\underline{\sigma}, j'\underline{\sigma}'}^{-1} \psi_{\underline{\sigma}'}^z(y_{j'})$$

The generalized eigenfunctions of  $\hat{H}$  are

$$\Phi_{\gamma}^{\sigma}(\lambda, \omega) = \frac{(\lambda - \underline{\alpha} \underline{\sigma})^{\frac{1}{4}}}{4\pi^{\frac{3}{2}}} \left[ e^{i\sqrt{\lambda - \underline{\alpha} \underline{\sigma}} \omega \cdot} \otimes \mathcal{X}_{\underline{\sigma}} + \sum_{j', \underline{\sigma}', j} (\Gamma_{\gamma}(\lambda))_{j' \underline{\sigma}', j \underline{\sigma}}^{-1} e^{i\sqrt{\lambda - \underline{\alpha} \underline{\sigma}} \omega y_j} \frac{e^{i\sqrt{\lambda - \underline{\alpha} \underline{\sigma}'} |\cdot - y_{j'}|}}{4\pi |\cdot - y_{j'}|} \otimes \mathcal{X}_{\underline{\sigma}'} \right] \quad \lambda \geq \underline{\sigma} \underline{\alpha}$$

The explicit form of the generalized eigenfunctions suggests a kind of "Huygens principle": each scatterer becomes source of spherical waves with a phase  $e^{i\sqrt{\lambda - \underline{\alpha} \underline{\sigma}} \omega y_j}$ .

Amplitudes can be computed for  $N = 2$ , but only in particular asymptotic limit and in perturbation theory one can say something when  $N > 2$ .

## Simple geometries: an example

- Consider the initial state

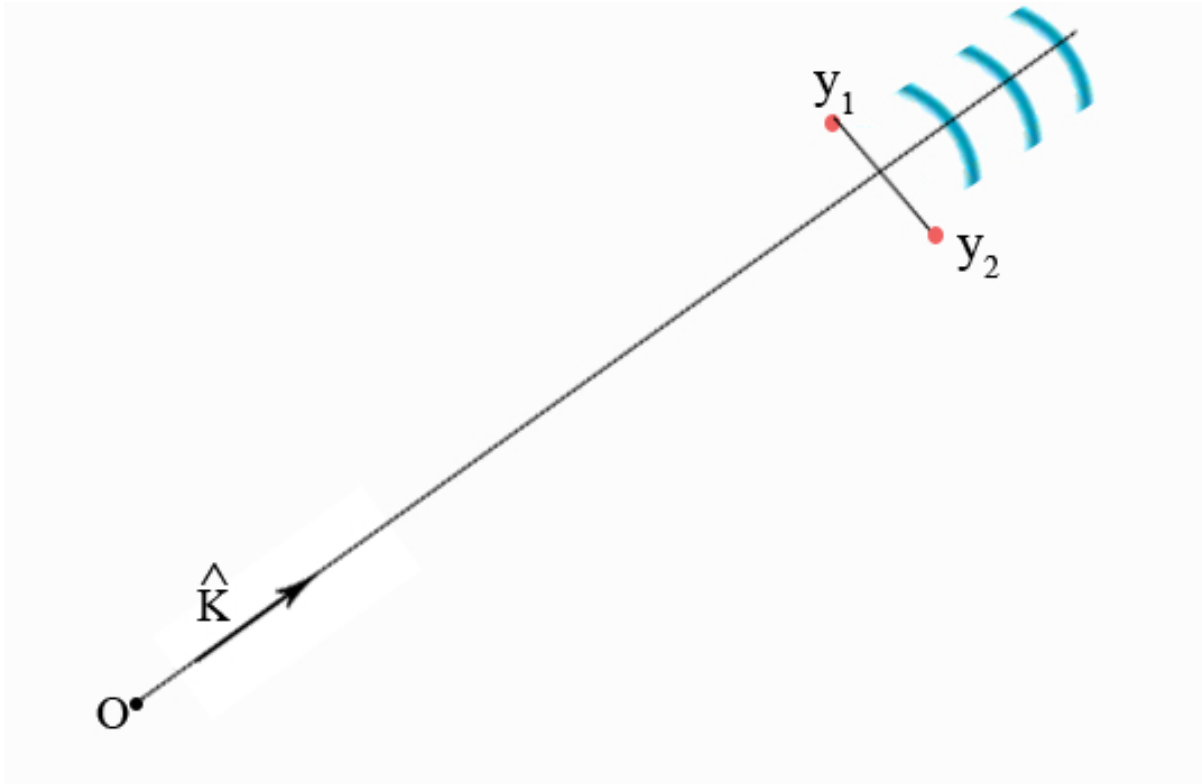
$$\Psi^0 = N \frac{e^{-\frac{|x|^2}{2s^2} + ik_0|x|}}{|x|} \otimes \mathcal{X}_{(-,-)} \quad N = \frac{1}{\sqrt{2s\pi^{\frac{3}{4}}}}; \quad \|\Psi^0\| = 1$$

$$\text{with } \frac{s}{|y_j|} \ll 1 \text{ and } \frac{1}{sk_0} \ll 1.$$

and the particular case:

$$|y_1| = |y_2| \quad \frac{|y_2 - y_1|}{y_i} \ll 1 \quad \frac{1}{k_0|y_2 - y_1|}$$

The probability that asymptotically both spins flip and the momentum has direction  $\hat{k}$  is maximal when  $\hat{k}$  is the perpendicular from the source to the segment connecting the two spins.



## Next steps

- a fully quantum one dimensional Coleman-Hepp model
  - the Huyghens principle for  $N > 2$
  - models of (non-interacting) particles measuring a spin
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