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# Percolation as a gauge theory I: general features and properties

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We review the formulation of the random percolation model interpreted as a fully consistent confining gauge theory, providing both theoretical motivations and the numerical results collected throughout the years. We show that, with a clever definition of the usual lattice pure gauge observables, the model well fits into the center vortex picture of confinement as well as agreeing with the effective string predictions. A particular emphasis is devoted to finite-temperature behaviour of Polyakov loop correlators, whose leading and next-to-leading quantum corrections are expected to be more-than-universal, in the sense that they are oblivious to the particular choice of the gauge group, leading to a very general prediction in the temperature-scaling of the string tension.

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## Lattice pure gauge theories

A pure gauge theory on the lattice usually:

- is constructed starting from the expectation values of **closed loops**;
- is probed by means of the **interquark potential** between static external sources;
- exhibits a **confinement/deconfinement phase transition**, where  $\sigma \rightarrow 0$ ;
- admits a **finite-temperature** reformulation with a deconfinement point  $T = T_c$ .

By choosing an action functional  $S$  and a gauge group  $G$ , the gluodynamics details are specified (conservation laws, fluxes sum rules, etc. . . ).

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## A general mechanism

The lattice allows for models with **discrete gauge group** (simpler to treat); moreover, we can switch to **(2+1)-dimensional systems** for simplicity and (in most cases) still observe confinement phenomenology.

Although each  $G$  determines its own universality class, it turns out that the confinement-related quantities usually vary very weakly. In particular, we consider the **universal ratio**

$$\frac{T_{cr}}{\sqrt{\sigma}} \ .$$

In order to better highlight the simple basic mechanism common to confining theories, we will proceed toward a dramatic **simplification** of the model.

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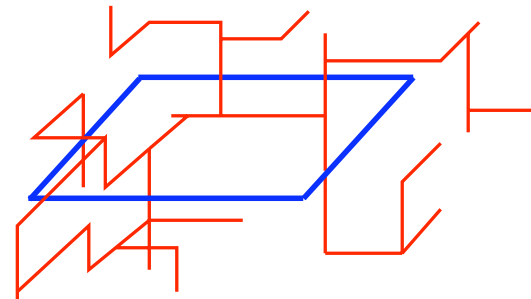
## Center vortices and confinement

In a given gauge configuration, there is a network of **center vortices**, extended objects related to nontrivial gauge transformations; a vortex gives a multiplicative contribution to  $W(R, T)$  if it **pierces the loop**, and there is confinement if the vortex graph is structured in such a way that  $W(R, T)$  decays with an area law (the vortices are able to disorder the gauge field).

Supporting facts:

- experimental observation of *center dominance*;
- sensitivity of finite-temperature observables to the *center* of  $\mathbb{G}$ .

Vortices must be *weakly* correlated.



When  $G = SU(N)$ , only  $N$ -uplets of vortices can join together (**flux conservation**): **dependence on gauge group!**

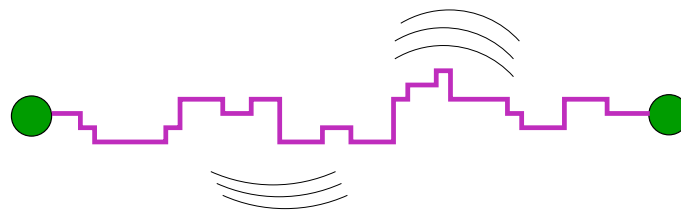
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## Flux tubes and confinement

The vacuum acts on the (chromo-)magnetic field as a **dual superconductor**, keeping all the flux between the sources squeezed in a **string-like tube** whose energy is proportional to its length  $\Rightarrow$  **linear growth of  $V(R)$** .

A strong coupling expansion of  $\langle W \rangle$  can be made in terms of string worldsheet surfaces (bounded by the loop contour); this expansion fails for too weak couplings.

Indeed, since the physics takes place in the so-called **rough phase** (the string **fluctuates quantistically on any length scale**), there are many all-new predictions that can be tested on the lattice.



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## The flux tube as a vibrating string

In the large interquark separation limit ( $R \rightarrow \infty$ ), the rough flux tube is treated as a **massless bosonic string**, with **Nambu-Goto action**  $\propto -(\sigma \cdot \text{Area})$ . The expansion parameter is indeed  $(\sigma \cdot \text{Area})^{-1} \ll 1$ .

This effective theory is consistent only for  $D = 26$ ; however, by a slight modification in the action, another effective theory was proposed [Polchinski, Strominger, 1991], which mimics the Nambu-Goto formulation but is valid in any dimensionality  $D \geq 3$ . Open-closed string duality arguments in  $D = 3$ , along with some calculations for any  $D$ , suggest **the first two terms of the effective theory to be universal**.

Note: the “class of actions” of the Nambu-Goto family unambiguously determines the functional form of a loop up to the NLO order: were it possible to prove the presence of a further term, **an actual realisation of an effective string theory would have been observed** (*more on effective strings later!*).

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## Toward percolation theory

Let's start with the  $3D \mathbb{Z}_2$  theory, the simplest nontrivial gauge model.

- Kramers-Wannier duality transformation  $\Rightarrow$   $3D$  Ising model (one-to-one mapping between observables and couplings in the two models).
- Fortuin-Kasteleyn reformulation for the Ising model, in terms of random clusters of aligned sites.
- In this context,  $W(R, T)$  is zero whenever a magnetised cluster is linked to the loop, since this fact is incompatible with the  $\mathbb{Z}_2$  unit flux running on the loop contour. To be precise, a generic even winding number is allowed (following  $\mathbb{Z}_2$  algebra).

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## Random percolation as a gauge theory

Naive, simple extension of the above recipe, in which only the **topological properties** are relevant:

- The lattice links are independently set to *on* or *off* according to probabilities  $p$  and  $1 - p$ : a **random cloud of connected *on* clusters** is formed.
- This is a **critical system**: there exist a value  $p_{cr}$  at which an **infinite connected network** appears (*percolation threshold, second order critical point*).
- Formally, this model has  $\mathbb{G} \equiv \{e\}$  and  $Z \equiv 1$ .
- We need observables sensitive only to the configuration topology (no *dangling ends* for instance), as some sort of **gauge invariance**.

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## Wilson loops and confinement in random percolation

One is led to define the Wilson loop value  $W(\mathcal{C})$  in a given configuration  $\mathcal{C}$  as:

$W = 1$  if no clusters are *linked* to the loop ;  $W = 0$  otherwise .

The definition is completely invariant under any transformation that does not alter the *loop* structure of the configuration, as required.

$p < p_{cr} \Rightarrow$  **finite-size clusters** give the perimeter law  $\Rightarrow$  **deconfinement**

$p > p_{cr} \Rightarrow$  the **infinite cluster** can pierce  $W$  in any point  $\Rightarrow$  **confinement**

This definition can be applied to Polyakov-Polyakov correlator functions in a straightforward manner.

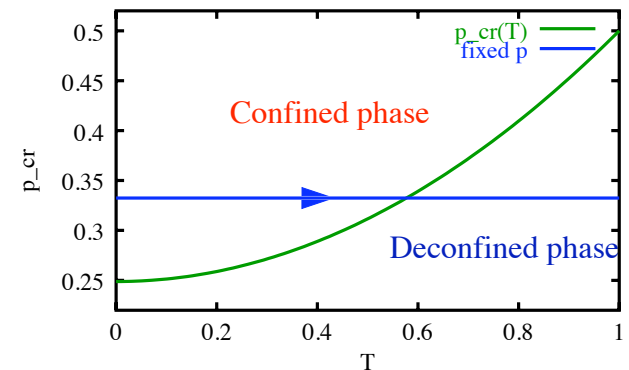
# Finite-temperature deconfinement in random percolation

At finite temperature  $T$ , the system is a 2D infinite slice with thickness  $\propto 1/T$ : the threshold probability  $p_{cr}(T)$  for the infinite cluster to appear is now a function of  $T$  and follows two-dimensional scaling laws.

Since  $p_{cr}(T)$  is an increasing function of  $T$ , by keeping a fixed probability  $p_{cr}(0) < p < p_{cr}(\infty)$  and heating up the system, the infinite cluster at some point  $T_{cr}$  vanishes away, leaving finite pieces that no longer give the area law to the loop behaviour: this is precisely a finite temperature deconfinement transition.

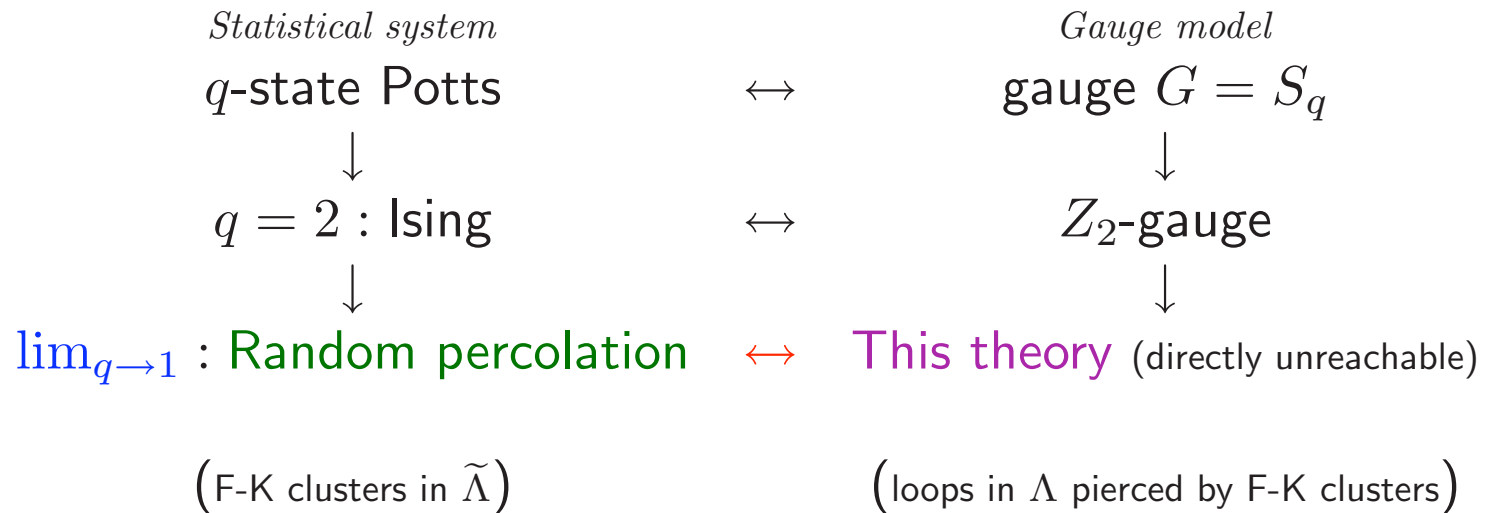
The amplitude ratio  $T_{cr}/\sqrt{\sigma}$  is well defined, with scaling laws:

$$\sigma(p) = S(p - p_{cr})^{2\nu} ;$$
$$T = \tau_0(p_{cr}(T) - p_{cr}^{3D})^\nu .$$



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# The gauge model is dual to the percolative system



The percolating graph stands for the F-K cluster structure; nevertheless, since any conservation law is dropped, it can be thought of as the center vortex network (which carries the “trivial group” charge).

**Numerical advantage:** with  $Z = 1$  (no update processes), high statistics are attainable.

**Conceptual advantage:** the core of confinement can be observed without group-specific artifacts.

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## Established numerical results

A consistent picture has emerged, which gives this theory the full status of a well-behaved gauge theory:

- Zero-temperature deconfinement transition, with well-defined string tension scaling.
- Glueball spectrum comparable to that of other gauge theories.
- Finite-temperature transition well studied via Polyakov-Polyakov correlators.
- Universality of the adimensional ratio  $T_c/\sqrt{\sigma} \simeq 1.5$ .

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## A quest for a string theory

Wilson and Polyakov loops clearly show the effects of an **underlying fluctuating string** correction, even though the model does not explicitly include such objects.

Example: For square Wilson loop at zero temperature, using the LO function  $W(R, R) \propto R^{1/4} \exp(-\sigma R^2 - 2cR)$ , instead of the “classical” form  $W(R, R) \propto \exp(-\sigma R^2 - 2cR)$ , leads to much better agreement with the numerical data and allows a fine determination of the string tension.

These corrections (to the “classical” area+perimeter law) are found to match, at the NLO, those from the Nambu-Goto predictions (*see the talk by P. Giudice*).

With the high statistics available, one can look for **extremely fine string effects** in loops’ functional form. This is particularly true for the finite-temperature Polyakov-Polyakov correlation function . . .

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## Predictions from the effective string theory

The Polyakov-Polyakov correlation function has the form of a “classical” term, with a multiplicative contribution coming from the rough string fluctuations:

$$\langle P(0)P^\dagger(R) \rangle_{T=1/L} \propto [\exp(-\sigma RL - cL)] \cdot Z_q .$$

With an order-by-order  $\zeta$ -function regularisation procedure, from the N-G action

$$S_{NG} = \sigma \int_0^L d\tau \int_0^R dr \sqrt{g} ,$$

it is possible to express the correction  $Z_q$  as an expansion (valid in the infrared limit) in the small quantity  $(\sigma RL)^{-1}$ .

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## NLO expansion for the Polyakov-Polyakov correlator

Consider a couple of Polyakov loops at a distance  $R$  in a system of temperature  $T = 1/L$  (in lattice spacings). At the Next-to-leading order in  $(\sigma RL)^{-1}$  the functional form obtained is:

$$\langle P(0)P^\dagger(R) \rangle = \frac{e^{-cL - \sigma RL - \frac{(D-2)\pi^2 L [2E_4(\tau) - E_2^2(\tau)]}{1152\sigma R^3}} + \mathcal{O}(1/R^5)}{\eta(\tau)^{D-2}},$$

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{Dedekind eta function} \quad \left( \tau = \frac{iL}{2R} ; \quad q(\tau) = e^{2\pi i \tau} = e^{-\pi \frac{L}{R}} \right)$$

$$E_2(\tau) \equiv 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \quad \text{second Eistenstein function}$$

$$E_4(\tau) \equiv 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \quad \text{fourth Eistenstein function}$$

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## String tension at finite temperature

The LO and NLO terms are *universal* in a much broader sense than the usual statistical meaning: they are expected from **any pure gauge confining theory** (for a given dimensionality) which flows to a massless bosonic effective string in the infrared.

By making use of the properties of the Eisenstein and Dedekind functions under modular transformation, the equation

$$\langle P(0)P^\dagger(R) \rangle = \frac{e^{-cL - \sigma RL - \frac{(D-2)\pi^2 L [2E_4(\tau) - E_2^2(\tau)]}{1152\sigma R^3} + \mathcal{O}(1/R^5)}}{\eta(\tau)^{D-2}},$$

can be recast in terms of **temperature-dependence of the string tension** as:

$$\sigma(L) = \sigma_0 - \frac{\pi}{6L^2} - \frac{\pi^2}{72\sigma_0 L^4}, \quad R \rightarrow \infty \quad (\text{NLO}),$$

where  $\sigma_0$  stands for the zero-temperature string tension.

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## Conclusions

Percolation is a **well-defined gauge theory**, notwithstanding having been constructed in **purely topological terms**; moreover, it can somewhat grab the common features of all confining theories and, at the same time, get rid of group-dependent dynamics details.

Having no dynamics at all, this theory is a gold mine in **computational terms** (no thermalisation, no critical slowing down, no update process) and allows to reach impressive statistics and volumes.

The key feature is the clear appearance of effects which can be ascribed to an **effective confining string**. These effects match the N-G prediction up to the **universality threshold**, showing that random percolation encodes an **actual realisation of a** (necessarily finite and predictive) **string model**, whose details will be contained in the next-to-NLO terms.