

A classification of the supersymmetric solutions of $N=4$ SYM theory

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Based on a work in progress with F. Galli, D. Klemm and C. Pedroli

- 1 EUCLIDEAN $N = 4$ SYM IN 4 DIMENSIONS AND INSTANTONS
- 2 SPINORIAL GEOMETRY
- 3 CLASSIFICATION OF SUSY SOLUTIONS OF $N = 4$ SYM
- 4 MORE

$N = 4$ SUPER YANG-MILLS

- Maximally supersymmetric gauge theory (no gravity) in 4D
- Prototype theory in gauge/gravity correspondences - AdS_5/CFT_4
- Euclidean theory obtained by dimensional reduction of 10D $N=1$ SYM on a $T^{1,5}$:
 $SO(1,9) \rightarrow SO(4)$ (space-time) \times $SO(1,5)$ (R-symmetry)
- Field content : A_μ (gauge fields), $\lambda^{\alpha,A}$ and $\bar{\lambda}_{\dot{\alpha},A}$ (Weyl spinors), $\phi^{AB} = \phi^{[AB]}$ (scalar fields), $A, B = 1, \dots, 4$, $\alpha, \dot{\alpha} = 1, 2$. All fields are in the adjoint rep of gauge group.

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Besides the global space-time and internal symmetries $SO(4)$ and $SO(1, 5)$, the theory is also invariant under the following supersymmetry transformations (with parameters $\bar{\xi}_A^{\dot{\alpha}}$ and ξ_α^A) :

$$\begin{cases} \delta A_\mu = -i\bar{\xi}_A^{\dot{\alpha}}\bar{\sigma}_{\mu\dot{\alpha}\beta}\lambda^{\beta,A} + i\bar{\lambda}_{\dot{\beta},A}\sigma_\mu^{\alpha\dot{\beta}}\xi_\alpha^A \\ \delta\phi^{AB} = \sqrt{2}\left(\xi^{\alpha,A}\lambda_\alpha^B - \xi^{\alpha,B}\lambda_\alpha^A + \epsilon^{ABCD}\bar{\xi}_C^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha},D}\right) \\ \delta\lambda^{\alpha,A} = -\frac{1}{2}\sigma^{\mu\nu\alpha}{}_\beta F_{\mu\nu}\xi^{\beta,A} - i\sqrt{2}\bar{\xi}_{\dot{\alpha},B}\mathcal{D}^{\alpha\dot{\alpha}}\phi^{AB} + [\phi^{AB}, \bar{\phi}_{BC}]\xi^{\alpha,C} \\ \delta\bar{\lambda}_{\dot{\alpha},A} = -\frac{1}{2}\bar{\sigma}^{\mu\nu}{}_{\dot{\alpha}}{}^\beta F_{\mu\nu}\bar{\xi}_{\dot{\beta},A} + i\sqrt{2}\xi^{\alpha,B}\mathcal{D}_{\dot{\alpha}\alpha}\bar{\phi}_{AB} + [\bar{\phi}_{AB}, \phi^{BC}]\bar{\xi}_{\dot{\alpha},C}. \end{cases}$$

The matrices $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ generating the two inequivalent representations of $\text{Spin}(4)$ are (anti-)selfdual :

$$\sigma_{\mu\nu} = -(\star\sigma)_{\mu\nu} \quad \bar{\sigma}_{\mu\nu} = (\star\bar{\sigma})_{\mu\nu}$$



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Yang-Mills instantons are solutions of the classical euclidean equations of motion with **finite** action.

They are classified into sectors characterized by an integer number k , the **instanton number**.

In a given topological sector, the solution to the field equations that minimizes the action has either **selfdual** or **anti-selfdual** field strength :

$$F_{\mu\nu} = \pm(\star F)_{\mu\nu}$$

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An selfdual configuration (with other fields vanishing) will therefore automatically represent a supersymmetric solution, since for $F_{\mu\nu} = (\star F)_{\mu\nu}$ the susy variation

$$\delta\lambda^{\alpha,A} = -\frac{1}{2}\sigma^{\mu\nu\alpha}{}_{\beta}F_{\mu\nu}\xi^{\beta,A}$$

vanishes for any susy parameter $\xi^{\beta,A}$, from the anti-selfduality of $\sigma^{\mu\nu}$.

Basic idea : represent **spinors** as **differential forms**

Lawson, Michelsohn/Gillard, Gran, Papadopoulos

The space of Dirac spinors of $\text{Spin}(4)$ is represented as $\Delta_{\mathbb{C}} = \Lambda^*(\mathbb{R}^2 \otimes \mathbb{C})$, with $\mathbb{R}^2 =: \text{span}(e_1, e_2)$.

This is nothing other than the space spanned by $\{1, e_1, e_2, e_1 \wedge e_2 \equiv e_{12}\}$.

That is, a Dirac spinor is written as

$$\lambda = \lambda_0 1 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_{12} e_{12}$$

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The action of the 4 gamma matrices on $\Delta_{\mathbb{C}}$ is given by ($i = 1, 2$) :

$$\gamma_i \lambda = e_i \wedge \lambda + e_i \lrcorner \lambda \quad \gamma_{i+2} \lambda = i e_i \wedge \lambda - i e_i \lrcorner \lambda$$

One checks that they indeed form a representation of the Clifford algebra on $\Delta_{\mathbb{C}}$: $\{\gamma_\mu, \gamma_\nu\} \lambda = 2\delta_{\mu\nu} \lambda$

With the chirality operator $\gamma^5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$, one observes that the even forms 1 and e_{12} have positive chirality ($\gamma^5 \eta = +\eta$) while e_1, e_2 have negative chirality ($\gamma^5 \eta = -\eta$).

$$\Delta_{\mathbb{C}} = \Lambda^{even}(\mathbb{R}^2 \otimes \mathbb{C}) \oplus \Lambda^{odd}(\mathbb{R}^2 \otimes \mathbb{C})$$

Decomposition of a Dirac spinor in Weyl spinors of definite chiralities.

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Simplification 1 : Use a basis where the gamma matrices act like annihilation and creation operators

$$\begin{aligned}\Gamma_1 &= \frac{1}{\sqrt{2}} (\gamma_1 - i\gamma_3); & \Gamma_{\bar{1}} &= \frac{1}{\sqrt{2}} (\gamma_1 + i\gamma_3) \\ \Gamma_2 &= \frac{1}{\sqrt{2}} (\gamma_2 - i\gamma_4); & \Gamma_{\bar{2}} &= \frac{1}{\sqrt{2}} (\gamma_2 + i\gamma_4).\end{aligned}$$

The Clifford algebra becomes that of fermionic creation-annihilation operators : $\{\Gamma_1, \Gamma_{\bar{1}}\} = 2$, $\{\Gamma_2, \Gamma_{\bar{2}}\} = 2$.

The spinor **1** is a **Clifford vacuum** ($\Gamma_{\bar{1}}1 = \Gamma_{\bar{2}}1 = 0$) from which $\Delta_{\mathbb{C}}$ can be reconstructed.

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SPINORIAL GEOMETRY

Advantage :

	1	e_1	e_2	e_{12}
Γ_1	$\sqrt{2}e_1$	0	$\sqrt{2}e_{12}$	0
$\Gamma_{\bar{1}}$	0	$\sqrt{2}$	0	$\sqrt{2}e_2$
Γ_2	$\sqrt{2}e_2$	$-\sqrt{2}e_{12}$	0	0
$\Gamma_{\bar{2}}$	0	0	$\sqrt{2}$	$-\sqrt{2}e_1$
$\Gamma_{1\bar{1}}$	-1	e_1	$-e_2$	e_{12}
Γ_{12}	$2e_{12}$	0	0	0
$\Gamma_{1\bar{2}}$	0	0	$2e_1$	0
$\Gamma_{\bar{1}2}$	0	$-2e_2$	0	0
$\Gamma_{\bar{1}\bar{2}}$	0	0	0	-21
$\Gamma_{2\bar{2}}$	-1	$-e_1$	e_2	e_{12}

Lots of zeroes when acting on spinors with gamma matrices or Lorentz generators!

SUPERSYMMETRIC SOLUTIONS OF $N = 4$ SYM

Focus on purely bosonic configurations ($\lambda = \bar{\lambda} = 0$). Look for configurations $\{A_\mu, \phi^{AB}\}$ solving the equations of motion and satisfying

$$\begin{cases} -\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\xi^A + \sqrt{2}\Gamma_\mu\bar{\xi}_B D^\mu\phi^{AB} + [\phi^{AB}, \bar{\phi}_{BC}]\xi^C = 0 \\ -\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\bar{\xi}_A + \sqrt{2}\Gamma_\mu\xi^B D_\mu\bar{\phi}_{AB} + [\bar{\phi}_{AB}, \phi^{BC}]\bar{\xi}_C = 0. \end{cases}$$

for non-vanishing susy parameters ξ and $\bar{\xi}$.

Focus on positive-chirality Killing spinors (at most 8 real susys) :

$$\xi^{\alpha,A} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} 1 + \begin{pmatrix} -d^* \\ -c^* \\ b^* \\ a^* \end{pmatrix} e_{12}$$

Simplification 2 : Use the global space-time $SO(4)$ and internal $SO(1,5)$ symmetries to bring the first spinor to an easier form.

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SUPERSYMMETRIC SOLUTIONS OF $N = 4$ SYM

We can bring a general positive-chirality spinor to the form

$$\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e_{12} \quad \equiv \quad \begin{array}{|c|c|c|c|} \hline \bullet & & & \\ \hline \end{array}$$

The susy equations impose ($\phi^{AB} = 0 \leftrightarrow$ selfduality condition)

$$F^{1\bar{1}} + F^{2\bar{2}} - [\phi^{12}, \phi^{34}] + [\phi^{13}, \phi^{24}] = 0$$

$$F^{1\bar{2}} + [\phi^{24}, \phi^{34}] = 0$$

$$D^1 \phi^{34} - D^{\bar{2}} \phi^{13} = 0 \quad , \quad D^1 \phi^{24} - D^{\bar{2}} \phi^{12} = 0$$

$$[\phi^{23}, \phi^{AB}] = 0 \quad , \quad D^\mu \phi^{23} = 0;$$

Remark : these equations and the Bianchi identities automatically imply that the equations of motion for A_μ and ϕ^{AB} (except that for ϕ^{14}) are satisfied!

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$$D^1 \phi^{34} - D^{\bar{2}} \phi^{13} = 0 \quad , \quad D^1 \phi^{24} - D^{\bar{2}} \phi^{12} = 0$$

$$[\phi^{23}, \phi^{AB}] = 0 \quad , \quad D^\mu \phi^{23} = 0;$$

Remark : these equations and the Bianchi identities automatically imply that the equations of motion for A_μ and ϕ^{AB} (except that for ϕ^{14}) are satisfied!

SUPERSYMMETRIC SOLUTIONS OF $N = 4$ SYM

We can bring a general positive-chirality spinor to the form

$$\xi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e_{12} \quad \equiv \quad \begin{array}{|c|c|c|c|} \hline \bullet & & & \\ \hline \end{array}$$

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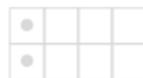
MORE SUSYS

Requiring the presence of more Killing spinors will yield additional constraints.

To bring the first Killing spinor to the previous form, we didn't make use of all of the global symmetries. There exists a stability subgroup which can be used to simplify the second Killing spinor.

We can bring the second Killing spinor either to

$$\xi_2 = \begin{pmatrix} a \\ 0 \\ 0 \\ d \end{pmatrix} 1 + \begin{pmatrix} -d^* \\ 0 \\ 0 \\ a^* \end{pmatrix} e_{12} \quad \text{or} \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} 1 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e_{12}$$



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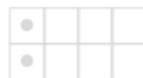
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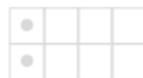
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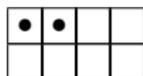
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SUPERSYMMETRIC SOLUTIONS OF $N = 4$ SYM :

MORE SUSYS

Pursuing the same reasoning, one notices that certain configurations do not exist.

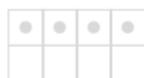
For example, the constraints coming from having 4 Killing spinors on the same $SO(4)$ orbit



imply that $F_{\mu\nu}$ is selfdual and that the scalar fields vanish. But then the susy variation reduces to

$$\delta\lambda^A = -\frac{1}{2}F^{\mu\nu}\sigma_{\mu\nu}\xi^A \quad (1)$$

and vanishes for any ξ^A .

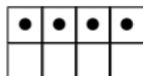


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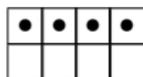


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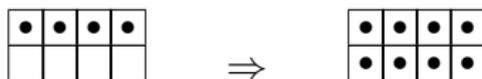
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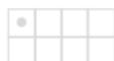
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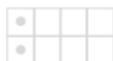
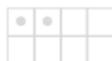
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CLASSIFICATION WITH POSITIVE-CHIRALITY KILLING SPINORS



One susy



Two susys



Three susys

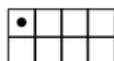


Four susys

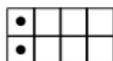
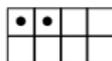


Eight susys : instanton

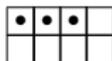
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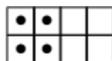
One susy



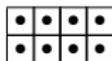
Two susys



Three susys



Four susys



Eight susys : instanton

MORE

- Classification can be extended to mixed-chirality Killing spinors
- Fermionic configurations a priori exist, but can have at most 7 supersymmetries (out of 16)
- Same analysis can be done for lorentzian $N = 4$ SYM

More :

- Find explicit solutions, clarify relationship between susy/selfdual configs/instantons)
- Match with susy solutions/instantons in IIB sugra
- Apply these techniques to classify solutions of other theories