

On Instantons and Donaldson–Thomas Invariants

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Geometry and Topological Theories

- Topological (string and field) theories have been extensively used in the last decades as simplified versions of physical models where non perturbative physics can be studied in a controllable setup.
- We will only consider topological theories that can be obtained via a twisting procedure. The so-called topological twist kills the dynamics of a physical model and leaves no propagating degree of freedom.
- This trick is very useful when we want to study the structure of the vacuum and how this is modified by non perturbative effects.
- For example in field theory one can check several predictions of non perturbative dualities. In string theory one gets informations on how geometry is modified at the quantum level.

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Topological Strings on a Calabi–Yau

- When the twisting procedure is applied to the superstring one obtains the topological string.
- A particularly interesting class of manifolds where this theory can be defined are the so-called Calabi–Yaus that play a role in phenomenological application of superstring theory.
- On these manifolds the topological string computes a certain class of observables that are related to geometrical invariants.
- Among these invariants are the Donaldson–Thomas. They count bound states of D6–D2–D0 branes (extended solitonic objects that arise in string theory).
- I will report on some progress to understand these invariants via a topological Yang–Mills theory that lives on the D6 branes, for local toric Calabi–Yaus

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Topological Yang–Mills in 6d

- Topological Yang–Mills in 6d has the form

$$\frac{1}{2} \int \text{Tr} \left(F \wedge F \wedge k_0 + \frac{\vartheta}{3} F \wedge F \wedge F \right) + \text{BRST-exact}$$

- The matter content of the theory is a gauge field A_μ , a Higgs field Φ a $(3,0)$ -form ρ plus twisted fermions.
- A consequence of the twisting procedure is that the semiclassical approximation becomes exact and the theory localizes on solutions of the Donaldson–Uhlenbeck–Yau equations

$$F^{(0,2)} = \bar{\partial}_A^\dagger \rho, \quad F^{(1,1)} \wedge k_0 \wedge k_0 + [\rho, \bar{\rho}] = 0, \quad d_A \Phi = 0$$

that parametrize holomorphic bundles: generalized instantons [Iqbal–Nekrasov–Okounkov–Vafa]

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Counting Instantons

- The partition function can be evaluated by using equivariant localization a la Nekrasov (as in $\mathcal{N} = 2$ super Yang–Mills in 4d) at least on \mathbb{C}^3 .
- This consists in a two step procedure:
- One considers a noncommutative deformation of the theory that resolves UV singularities associated to small instantons. This step can be understood as a regularization.
- Then one redefines the BRST operator as an equivariant differential on the space of field configurations. This allows the use of the powerful techniques of equivariant localization that reduce difficult integrals over the instanton moduli spaces to sums over critical points of the BRST charge.

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An ADHM formalism on \mathbb{C}^3

- We can explicitly show a direct link between the generalized instantons and Donaldson–Thomas invariants by parametrizing the moduli space of these invariants with a set of ADHM–like equations (at least on \mathbb{C}^3).
- The quantum mechanical model constructed around these equations describes quantization of the collective coordinates around each 6D instanton solution.
- In stringy language this corresponds to study the dynamics of a gas of D0 branes bound to the D6 branes.
- Finally one identifies the counting of the instanton solutions as an index associated with this quantum mechanical model.

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Toric Geometries

- These procedure can now be generalized to arbitrary toric varieties.
- Recall that a toric variety can be described in purely combinatorial terms. All the geometric information is encoded in a trivalent diagram: each vertex is a copy of \mathbb{C}^3 and each edge a bundle over \mathbb{P}^1 .
- We can evaluate the partition function of 6d Yang–Mills on a toric variety via a set of Feynman–like rules: the 'vertices' are given by the index computed by the ADHM quantum mechanics on \mathbb{C}^3 and the 'propagators' represent instantons localized on \mathbb{P}^1 .
- The results coincides with the topological string amplitude (as computed for example with the topological vertex).

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Conclusions and work in progress

- We have described a formalism that computes instantons in 6d Yang–Mills and clarifies their relation with the Donaldson–Thomas invariants on any toric variety
- This formalism is essentially a higher dimensional generalization of the ADHM construction
- Our formalism can be extended to singular variety such as orbifolds. There it can be used to study an higher dimensional version of the McKay correspondence, which relates physics on a singular manifold with physics on its resolution
- Still many details have to be filled...

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